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Implementing a weighted measure of multivariate spatial autocorrelation

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Abstract

Bavaud (2024) builds on and significantly broadens earlier work on measuring spatial autocorrelation, extending to multivariate settings and in particular regional weights. While measurement of spatial autocorrelation in multivariate data has been approached previously, the addition of regional weights is a major advance, as regions often differ in their contributions to global measures. Thus far, the proposed implementation described here involves dense matrices, as does much multivariate analysis. It is shown that the implementation largely reproduces the results presented in Bavaud (2024).

1 Introduction

When exploring and analyzing areal data, it is known that the definition of the areal (regional) observations may influence our results. The administrative boundaries used for data collection and aggregation are seldom chosen by the researcher. Some entities may be smaller or larger than others in area or population. Other regional structures, such as economic or social clusters, may be divided among several observations. Entitation subject to these conditions can lead to observations that are internally heterogeneous, spatially autocorrelated, and potentially heteroscedastic (Gelfand, 2010; Haining, 2010). While we cannot resolve the problem of internal heterogeneity, spatial autocorrelation can be addressed using well-known spatial statistics methods, leaving heteroscedasticity, here most likely stemming from the uneven weighting given to observations.

In Bavaud (2024), the importance of regional weights is brought into sharp focus, and measures and tests for multivariate (also univariate) spatial autocorrelation are proposed. In Bivand (2017), it was argued that modelling the Boston housing value data set without taking the number of houses used to calculate the median per spatial unit, varying from 5 to 3031, could and probably does lead to spurious inferences. So measures of spatial autocorrelation handling regional weights thoroughly are very welcome, and deserve to be implemented in software in order that they may be applied where appropriate. As I still maintain the **spdep** R package (Bivand, 2025), which provides a range of other measures of spatial autocorrelation, I chose to use R for coding the δ measure proposed in Bavaud (2024), and at this stage not to attempt to avoid the use of dense matrices.

For the remainder of this description of how the measures have been implemented, page, equation, table and figure numbers, given without a specific reference, refer to pages, equations, tables or figures in Bavaud (2024).

1.1 Data sets

The implementation of the δ measure and the methods for constructing spatial weights has been based on the data used in Bavaud (2024), that is, the toy data set provided there (p. 587), and data from the French presidential election in 2022 (pp. 589-590).

The French department boundaries (without Corsica and overseas territories) were taken from GADM, administrative level 2,[1] and simplified with a tolerance of 25m. The political data agree with the tabulation reported in Wikipedia[2], while the social data were provided by colleagues in Lausanne. The adjacency matrix also provided by colleagues in Lausanne agrees with that generated from the department boundaries.

For purposes of comparison, brief use will also be made of the Guerry data set (Anselin, 2019; Anselin & Li, 2020; Dray & Jombart, 2011; Friendly & Dray, 2023).

1 `https://geodata.ucdavis.edu/gadm/gadm4.1/gpkg/gadm41_FRA.gpkg`, accessed 19 Apr. 2025.

2 `https://en.wikipedia.org/wiki/2022_French_presidential_election`, accessed 19 Apr. 2025.

2 Regional weights

Weights may be used in statistical exploration and analysis when the observed entities are characterised by differences in their relative importance. These weights, termed *analytical*, *reliability* or *precision weights* are typically used when the observations are aggregate values constructed from differing numbers of aggregated components, such as numbers of inhabitants. Here we term these weights *regional weights*.

On page 576, the specification of the regional weights is given, requiring that the relative importance of each region be non-zero and that the regional weights sum to unity. In the 2022 French presidential election data set, the regional weights were the sums of valid votes cast for first round candidates by the regional aggregate unit, the department, divided by the sum of all valid votes cast for all mainland departments. The regional weights vary from 0.0014, (Lozère) to 0.0379, (Nord), as also seen on page 589. Uniform regional weights (p. 580), $f_i = 1/n$ where n is 94, are all 0.0106.

3 Dissimilarity matrices

Dissimilarity matrices are used in multivariate analysis to describe the relative differences between observations based on the values of the observed variables. The dissimilarity matrix used here can be constructed both for the univariate and multivariate settings; the basic specification is that $d_{ij} \geq 0$, $d_{ij} = d_{ji}$ and $d_{ii} = 0$ (Eq. 11, p. 579); the matrix should be symmetric with zeros on the leading diagonal and off-diagonal elements non-negative. Here, the term *dissimilarity matrix* is used, rather than *feature dissimilarity matrix*, as the term *feature* may mean *variable* in data science, but means *spatial entity* in much of the geospatial literature. The dissimilarity matrix is constructed from the one or more variables whose joint spatial autocorrelation is being analysed. So far, the variables used for calculating the dissimilarities between observations have been assumed to be numeric rather than categorical or a mixture of numeric and categorical.

Construction of dissimilarity matrices in multivariate analysis with numerical variables is to use Euclidean or equivalently squared Euclidean distance: $d_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|^2 = \sum_{k=1}^{p}(x_{ik} - x_{jk})^2$ (for p variables of interest, pages 579-580 and equation 35, $\mathbf{D}$ is dense by construction), this is used for two of the French 2022 data set examples, the share of the first round vote achieved by Emmanuel Macron (data set $\mathbf{x}$), and the five social variables (data set $\mathbf{Y}$). As Anselin (2019, p. 138) points out, this is directly associated with Geary's c; Anselin (2019) also suggests that the scaling of variables of interest to mean zero and unity variance is advisable.

On page 586, extension to a squared Euclidean constructed as a chi-square dissimilarity is discussed for the vote counts for all $m = 12$ candidates in the first round of the 2022 election (data set $\mathbf{X}$). We know that the regional weights have been taken as the proportion of total votes cast in each department:

$$f_i = \frac{\sum_{k=1}^{m} x_{ik}}{\sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk}}$$

The dissimilarity for votes cast by candidate is then taken as the sum over the m candidates of the difference between the vote share of the candidate between departments multiplied by the inverse of the total vote share of that candidate (equation 34, rather suggestive of shift-share analysis, e.g. Bivand, 1999):

$$d_{ij} = \sum_{k=1}^{m} \frac{\sum_{h=1}^{n} \sum_{l=1}^{m} x_{hl}}{\sum_{h=1}^{n} x_{hk}} \left(\frac{x_{ik}}{\sum_{l=1}^{m} x_{il}} - \frac{x_{jk}}{\sum_{l=1}^{m} x_{jl}} \right)^2$$

4 Constructing spatial weights

When observations may be dependent on each other in multi-level designs, time series or by being proximate neighbours in space, their mutual dependency may be expressed as a matrix or equivalently a graph. In multi-level designs, the relationships between say pupils in a school class are shown as a block-diagonal matrix, where all pupils in a class are related to each other, but pupils in different classes are unrelated - typically the principal diagonal is zero. In the time series case, the first sub-diagonal represents the first-lag relationship, between time t and $t - 1$. Cliff & Ord (1973) systematized the relationships of proximate neighbours in space, shown as unity in the binary spatial weights matrix where observation i is a neighbour of observation j, and zero otherwise.

The introduction of regional weights means that the construction of spatial weights needs to be adjusted. Bavaud (1998) pointed out that row-standardised spatial weights constitute the conditional probability of visiting j straight from i. Such row-standardised spatial weights are frequently used, and the elements of the spatial weights matrix are set with rows summing to unity. All the regions in the adjacency matrix must be able to communicate with each other; no islands or sub-graphs are permitted. On page 577, it is stressed that the chosen regional weights (all non-zero and summing to unity) should constitute the unique stationary distribution of the weight-compatible (adjusted) spatial weights. Equation 2 on page 577 defines adjusted spatial weights as row-standardised, weight-compatible and reversible, if they satisfy:

$$\mathbf{W} \geq 0, \mathbf{W1} = \mathbf{1}, \mathbf{W}^{\top}\mathbf{f} = \mathbf{f}, \mathbf{\Pi W} = \mathbf{W}^{\top}\mathbf{\Pi}, \mathbf{\Pi} = \mathrm{diag}(\mathbf{f})$$

where $\mathbf{f}$ is a vector of regional weights, $\mathbf{W}$ are the adjusted spatial weights, and $\mathrm{diag}()$ is an operator creating a diagonal matrix with the given vector on the principal diagonal; $\mathbf{1}$ is an n-vector of ones.

Definition 1 on page 579 presents three ways of constructing adjusted spatial weights, followed by definition 2 on page 589 showing how to adjust a matrix of distances to meet the same criteria. At this stage, it is not obvious which method of adjustment should be chosen by the analyst. These construction methods will be described next, taking $\mathbf{A}$ as a symmetric binary adjacency matrix and $\mathbf{f}$ is a vector of regional weights.

In the proposed implementation in R, standard linear algebra is used, vectorizing all operations. For comparison, the adjacency matrix $\mathbf{A}$ for the 94 French departments has $94 \times 94 = 8836$ elements, of which 238 are non-zero.

4.1 Linearised diffusive weights

Linearised diffusive weights are discussed on page 578. The adjusted spatial weights constructed in this way also require a coefficient t:

$0 < t \leq t_1$ where $t_1 = \min(\mathbf{f}/\mathbf{r})$ and $\mathbf{r} = \mathbf{A1}$ ($\mathbf{r}$ are the row sums of $\mathbf{A}$, p. 579). Here (Eq. 7):

$$\mathbf{W}_{\text{ldw}} = \mathbf{I} - t(\mathbf{\Pi}^{-1}(\text{diag}(\mathbf{r}) - \mathbf{A}))$$

where $\mathbf{I} = \text{diag}(\mathbf{1})$ is the identity matrix.

The coefficient t may also take the value $t_2 = 1/\max(\xi)$, where ξ are the eigenvalues of the adjusted Laplacian matrix given in equation 7: $\mathbf{\Pi}^{-\frac{1}{2}}(\text{diag}(\mathbf{r}) - \mathbf{A})\mathbf{\Pi}^{-\frac{1}{2}}$ (p. 583). The count of non-zero elements in $\mathbf{W}_{\text{ldw}}$ for the French departments is 570. The Laplacian term: $\text{diag}(r) - \mathbf{A}$ corresponds to the structure matrix used in the intrinsic conditional autoregressive (ICAR) model (Held & Rue, 2010, p. 209).

4.2 Metropolis–Hastings weights

Starting from the row-standardized adjacencies $\mathbf{P} = \text{diag}(1/\mathbf{r})\mathbf{A}$, also termed the natural random walk transition matrix (p. 578), regional weights are introduced: $\mathbf{Q} = \mathbf{\Pi P}$ and the by-element minima, here taken as $\text{pmin}()$, of $\mathbf{Q}$ and its transpose are found: $\mathbf{\Gamma} = \text{pmin}(\mathbf{Q}, \mathbf{Q}^\top)$, with row sums $\gamma = \mathbf{\Gamma 1}$. Then the Metropolis–Hastings weights are (Eq. 9):

$$\mathbf{W}_{\text{mhw}} = \mathbf{\Pi}^{-1}\mathbf{E}, \mathbf{E} = \text{diag}(\mathbf{f} - \gamma) + \mathbf{\Gamma}$$

Here, as in the case of linearised diffusive weights, the principal diagonal and some other elements are non-zero, but the count of non-zero elements remains small at 562.

4.3 Iteratively fitted weights

Iteratively fitted weights first add g, a very small positive value, to the binary adjacency matrix $\mathbf{A}$, then employ iterative proportional fitting to iterate the weights matrix $\mathbf{W}_{\text{ifw}}$ to row and column margins very close to the regional weights $\mathbf{f}$. In this implementation, the `Ipfp` function from the **mipfp** package is used (Barthélemy & Suesse, 2018). As g is added to input zero elements, the output matrix $\mathbf{W}_{\text{ifw}}$ is dense, with, for the French departments, 8836 non-zero elements.

4.4 Graph distance weights

While the definition on page 589 of geographic distance weights is sufficiently clear, there is a minor discrepancy between the graph distance weights described on the same page with a maximum number of edges between the most distant nodes of 11, and that found from the graph distances found from the French department boundaries used for reproduction here. Using the `distances` function in the **igraph** package (Csárdi et al., 2024), the maximum number of edges between the most distant nodes is 12, with one most distant pair of departments being Pas-de-Calais–Pyrénées-Orientales.

Here a coefficient c is needed, where $c \in (0, c_1]$ and $c_1 = \frac{-1}{\min(\mathbf{B})}$. Taking $\mathbf{\Delta}$ as the symmetric matrix of numbers of edges separating nodes, which is by definition dense, we generate the adjusted spatial weights as (Eq. 33, p. 589):

$$\mathbf{W}_{\mathrm{gdw}} = \mathbf{1}\mathbf{1}^{\top} + (c\mathbf{B})\mathbf{\Pi}, \mathbf{B} = -\frac{1}{2}\mathbf{H}\mathbf{\Delta}\mathbf{H}^{\top}, \mathbf{H} = \mathbf{I} - \mathbf{1}\mathbf{f}^{\top}$$

The output matrix $\mathbf{W}_{\mathrm{gdw}}$ is dense, with, for the French departments, 8835 non-zero elements.

5 Implementing Bavaud's δ and comparison with other measures

As Bavaud's δ permits the analysis of spatial autocorrelation with possibly non-uniform regional weights for multivariate data, the three main arguments – the dissimilarity matrix $\mathbf{D}$, the adjusted spatial weights matrix $\mathbf{W}$, and the vector of regional weights $\mathbf{f}$ – must be prepared first. In this implementation, the regional weights $\mathbf{f}$ may be taken from the $\mathbf{W}$ object, as the constructor functions store the regional weights used as an attribute of their output matrices, in order to attempt to ensure consistency.

Starting from equation 1 on page 576, and equation 26 on page 586, Bavaud's δ is:

$$\delta = \frac{\mathrm{tr}(\mathbf{K_D}\mathbf{K_W})}{\mathrm{tr}(\mathbf{K_D})}$$

	$\mathbf{W}_a$		$\mathbf{W}_b$		$\mathbf{W}_c$	
	δ	z	δ	z	δ	z
X	0.96	10.51	0.75	15.59	0.55	14.21
Y	0.94	7.78	0.67	10.88	0.42	8.94
x	0.95	5.64	0.66	7.14	0.45	6.64

TABLE 1 – Values of Bavaud's autocorrelation index δ, together with its standardized value z.

where $\mathrm{tr}()$ is the trace of the given matrix (the sum of the principal diagonal), $\mathbf{K_D}$ is the dissimilarity kernel expressing differences in the relative position of observations in the space of the variables, and $\mathbf{K_W}$ is the spatial kernel expressing differences in the relative position of observations in geographical space. The kernels are (Eq. 1, p. 576 and Eq. 16, p. 581):

$$\mathbf{K_D} = -\frac{1}{2}\mathbf{\Pi}^{\frac{1}{2}}\mathbf{HDH\Pi}^{\frac{1}{2}}, \mathbf{H} = \mathbf{I} - \mathbf{1f}^\top, \mathbf{\Pi} = \mathrm{diag}(\mathbf{f})$$

and (Eq. 1, p. 576 and Eq. 21, p. 583):

$$\mathbf{K_W} = \mathbf{\Pi}^{\frac{1}{2}}\mathbf{W\Pi}^{-\frac{1}{2}} - \mathbf{f}^{\frac{1}{2}}(\mathbf{f}^{\frac{1}{2}})^\top$$

Using the toy example (pp. 586-587) and the proposed implementation of δ, we find that the output gives $\delta_A = 0.13$, $\delta_B = 0.57$, and $\delta_C = -1.00$, reproducing the original results. The standard deviates, using the moments of δ given on page 591, theorem 4, are: $z_A = 1.98$, $z_B = 1.98$, and $z_C = -2.24$, where the first two agree, but the latter does not; its probability value however does agree, $p_C = 0.025$, so z_C may be a misprint.

Next, we will try to reproduce the tabular results for the three data sets observed over the French departments; results for graph distance weights are omitted because the maximum edge counts across the graph were found to differ, so they would not be expected to agree. Table 1 reproduces the results in the original table 1 on page 582. Tables 2

	$\bar{\lambda}$	ν	κ	$\alpha(\lambda)$	$\gamma(\lambda)$
X	0.00073	3.07	0.32	7.25	56.65
Y	0.59738	2.14	0.46	7.54	58.91
x	0.00002	1.00	1.00	9.49	88.01

TABLE 2 – Spectral moments and other quantities for data sets.

	$\bar{\mu}$	Var(I)	$\alpha(\mu)$	$\gamma(\mu)$
$\mathbf{W}_a$	0.7920	0.00078	-1.54	2.79
$\mathbf{W}_b$	0.2504	0.00322	0.43	-1.10
$\mathbf{W}_c$	-0.0086	0.00486	0.28	-0.79

TABLE 3 – Spectral moments and other quantities for adjusted spatial weights.

and 3 reproduce those of both parts of the original table 2 on page 594.

Tables 4 and 5 to a certain extent reproduce the original tables 3 and 4 on page 595. Note that the values of $A(\delta)$ diverge slightly from those on page 595, as consequently do the values of p_{CF}, the Cornish-Fisher correction (Eq. 45, p. 593). In addition, the values of p_{CF} are not reported for $\mathbf{W}_b$ because the values of skewness and excess kurtosis do not fall in the domain given by Amédée-Manesme et al. (2019, p. 446, Eq. 24). The difference for p_{normal} for $\mathbf{W}_a$ and $\mathbf{Y}$ may be a difference in rounding or a misprint.

So far, the prototype implementation seems to be able to reproduce the original published output except for the skewness term, the Cornish-

	$\mathbf{W}_a$		$\mathbf{W}_b$		$\mathbf{W}_c$	
	$A(\delta)$	$\Gamma(\delta)$	$A(\delta)$	$\Gamma(\delta)$	$A(\delta)$	$\Gamma(\delta)$
X	-0.335	0.289	0.0942	-0.00952	0.0607	0.0142
Y	-0.348	0.301	0.0979	-0.00984	0.063	0.0148
x	-0.438	0.447	0.1232	-0.01396	0.0794	0.0226

TABLE 4 – Expected skewness $A(\delta)$ and excess kurtosis $\Gamma(\delta)$.

	$\mathbf{W}_a$		$\mathbf{W}_b$	$\mathbf{W}_c$	
	p_{normal}	p_{CF}	p_{normal}	p_{normal}	p_{CF}
$\mathbf{X}$	$4 \cdot 10^{-26}$	$4 \cdot 10^{-24}$	$4 \cdot 10^{-55}$	$4 \cdot 10^{-46}$	$7 \cdot 10^{-29}$
$\mathbf{Y}$	$3 \cdot 10^{-15}$	$2 \cdot 10^{-18}$	$7 \cdot 10^{-28}$	$2 \cdot 10^{-19}$	$2 \cdot 10^{-15}$
$\mathbf{x}$	$8 \cdot 10^{-09}$	$2 \cdot 10^{-11}$	$5 \cdot 10^{-13}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-09}$

TABLE 5 – One-tailed significance test of δ.

Fisher correction with very large standard deviate values, and possibly the graph distance method of constructing adjusted spatial weights.

5.1 Relationships to Moran's I

Equation 14 on page 580 asserts that δ will equal Moran's I with uniform regional weights in the univariate spatial case, tested here for row-standardized adjacencies and the vote share recorded for Emmanuel Macron by department, with squared Euclidean dissimilarities used as the dissimilarity measure. The value of Moran's I is 0.54376, equalling that of δ: 0.54376.

6 Methods for δ

Beyond the simple `print` and `summary` methods for the object returned by `spatialdelta`, the prototype implementation of δ in a forthcoming version of the R package **spdep**, several other kinds of method seemed warranted. These include `plot_spatialcoords` (Fig. 1), `plot_moran` (Figs. 2 and 3), `plot_spatialscree` (Fig. 8), `plot_factorialcoords` (Fig. 5), `plot_factorialscree` (Fig. 7), `localdelta` (Eq. 30) and `cornish_fisher` (Eq. 45).

6.1 Plots

Figure 1 reproduces the upper two panels of original figure 1, with the points proportional in size to the regional weights $\mathbf{f}$ scaled to match the range of the axes. Figure 2 shows the scree plots for $\mathbf{W}_a$, $\mathbf{W}_b$ and $\mathbf{W}_c$, as in original figure 8. Finally, figure 3 shows the Moran plots for Macron vote share for $\mathbf{W}_a$, $\mathbf{W}_b$ and $\mathbf{W}_c$; the slope coefficients shown in the x-axis label do not match δ exactly as asserted in the caption to original figure 3.

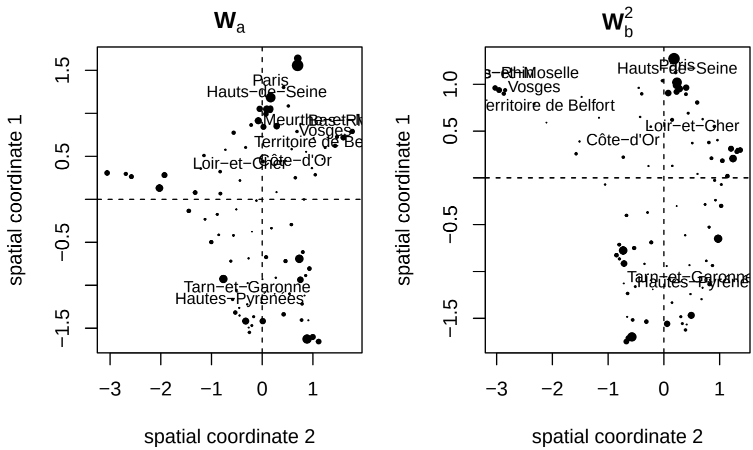

FIGURE 1 – Reproducing the upper row of original figure 1: spatial coordinates of $\mathbf{W}_a$ and $\mathbf{W}_b^2$.

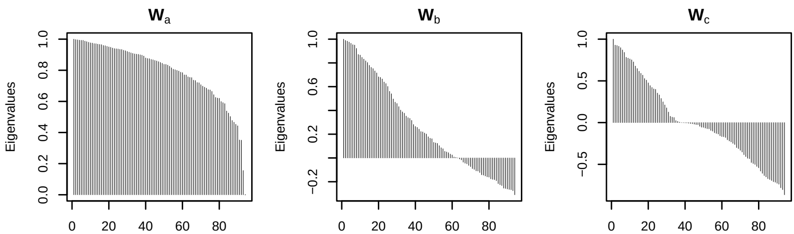

FIGURE 2 – Reproducing original figure 8: spatial scree plots.

6.2 Local δ

Local δ_i as defined in equation 30 on page 588 can be shown to equal local Moran's I_i in the Macron vote share univariate case (using squared Euclidean dissimilarities), with row-standardized adjacencies and uniform regional weights, as shown in figure 4 rendered with the **mapsf** package (Giraud, 2024). The relationship $\delta = \sum_{i=1}^{n} f_i \delta_i$ can also be

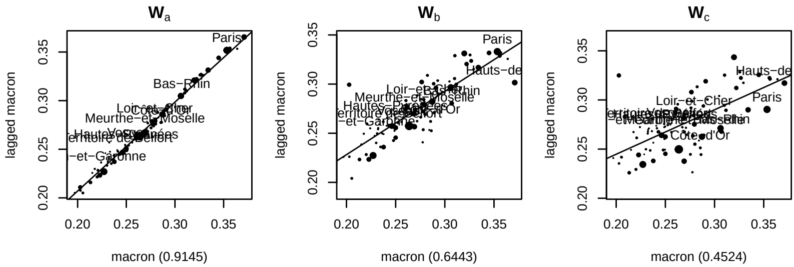

FIGURE 3 – Reproducing original figure 3: Moran-Anselin plots for Macron vote share.

shown to hold. The relationship between multivariate local δ_i and local Geary's c_i remains to be studied; figure 5 shows δ_i and local Geary's c_i for the five social variables $\mathbf{Y}$, row-standardized adjacencies and uniform regional weights.

7 Comparisons using the Guerry data set

Several explorations of spatial autocorrelation with multivariate data have used the French 85-department data set omitting Corsica, specifically six moral variables (Anselin, 2019; Anselin & Li, 2020; Dray & Jombart, 2011; Friendly & Dray, 2023).

Figures 6 and 7 show respectively the first two principal component scores of the scaled moral variables, corresponding to Anselin (2019, Figs. 1 and 2, p. 142) and Dray & Jombart (2011, Fig. 4, p. 2286) (signs of principal components flipped) which both scale the moral variables, and the first two factor coordinates of Metropolis–Hastings spatial contiguity weights of the scaled moral variables with population proportion regional weights. There is no need to show the first two factor coordinates of Metropolis–Hastings spatial contiguity weights with uniform regional weights because at least the first two are identical to the first two principal component scores of the scaled moral variables. Using population proportion regional weights, the first vector of factor coordinates is very similar to the first principal component scores, with larger differences for the second.

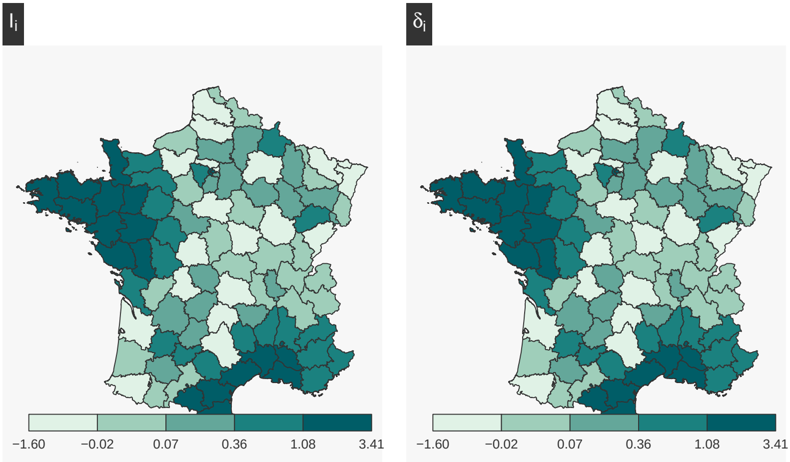

FIGURE 4 – Macron vote share with uniform regional weights and row-standardized adjacencies. Left: local Moran's I_i. Right: local δ_i.

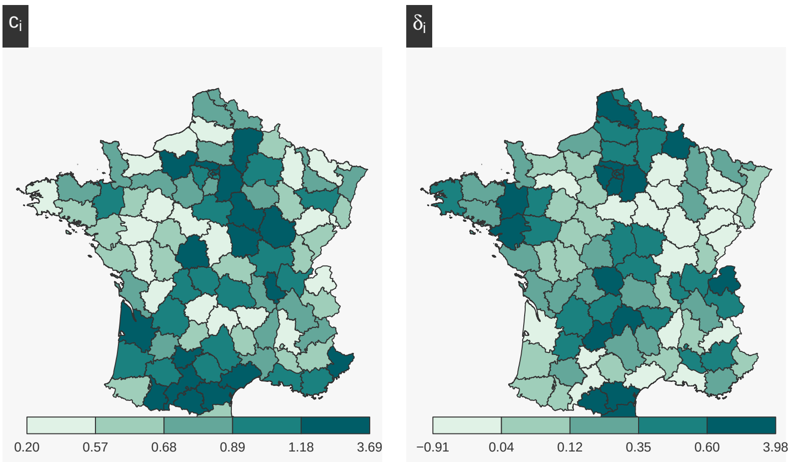

FIGURE 5 – Multivariate social data set with uniform regional weights and row-standardized adjacencies. Left: local Geary's c_i. Right: local δ_i.

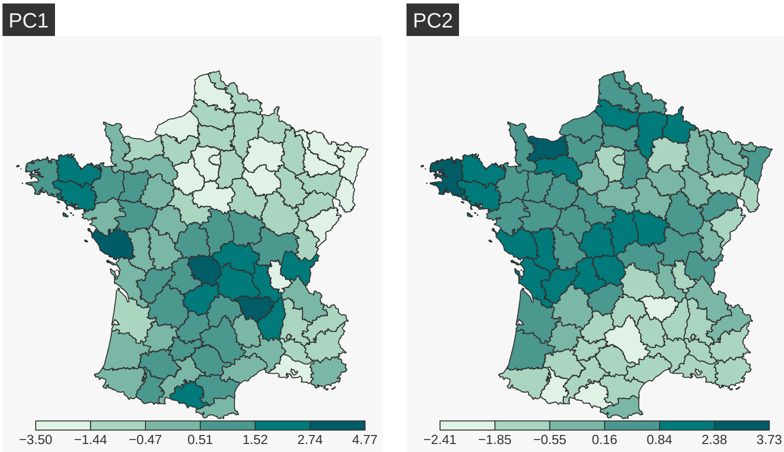

FIGURE 6 – Principal components for scaled moral variables. Left: PC1. Right: PC2.

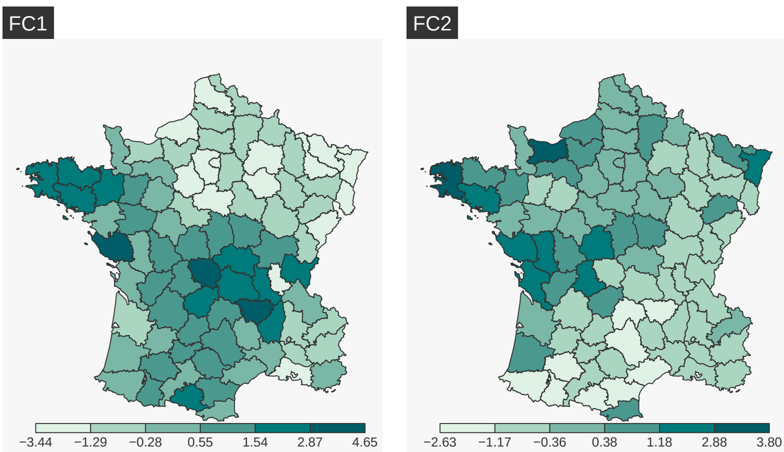

FIGURE 7 – Factorial coordinates for population weighted moral variables. Left: first factorial coordinate. Right: second factorial coordinate.

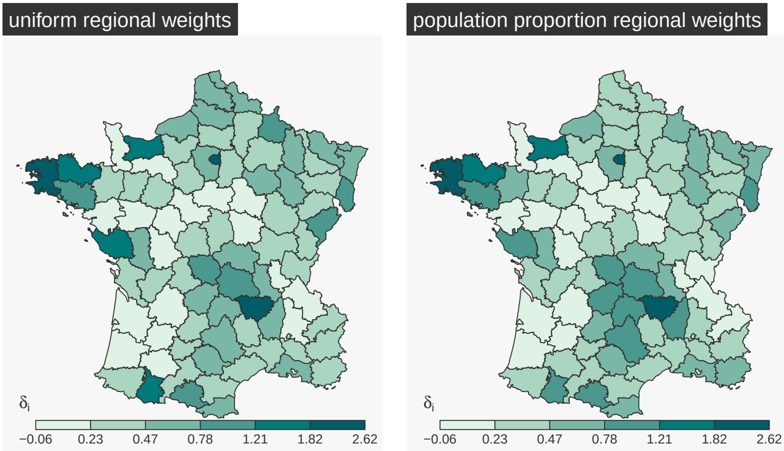

FIGURE 8 – Local δ_i values. Left: uniform regional weights. Right: population proportion regional weights.

Figure 8 shows the local δ_i values for Metropolis–Hastings spatial contiguity weights of the scaled moral variables for uniform and population proportion regional weights; here they are quite strongly correlated, but the population sizes only differ by less than an order of magnitude.

Finally, the left panel of figure 9 reproduces Anselin & Li (2020, Fig. 6, p. 502), while the right panel reproduces Anselin (2019, Fig. 11, p. 146), both approximately, as the `localC` implementation in **spdep** uses conditional permutation to calculate the pseudo-significance of local measures. Here it may well be the case that adjacency definitions vary, as Anselin (2019, footnote 6, p. 148) uses Queen contiguities, but Anselin & Li (2020, p. 496) appear to use $k = 6$ nearest neighbours, while Queen contiguities have been used here.

8 Conclusions

As reviewed in Bavaud (2024), there have been a number of suggestions going back several decades to measure spatial autocorrelation in multivariate settings. There have been some more serious studies, largely by the author himself, into the proper treatment of spatial weights, placing

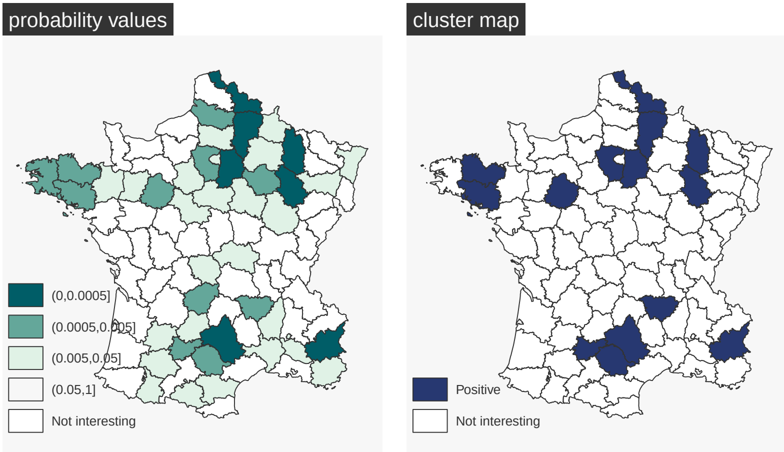

FIGURE 9 – Local c_i results. Left: classified probability values. Right: multivariate local Geary clusters.

them in the context of broader models showing how information may move between regional entities. The introduction of δ and local δ_i brings these two concerns with relationships between observations in variable space and in geographical space to a shared expression also taking into account regional weights. The implementation proposed here does need to be checked through broader use, but should assist in providing analysts with ways of exploring their data using these innovative and suggestive methods.

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